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We demonstrate the existence of a secular back-reaction on inflation using a simple scalar model. The model consists of a massless, minimally coupled scalar with a quartic self-interaction which is a spectator to  $\Lambda$ -driven inflation. To avoid problems with coincident propagators, and to make the scalars interact more like gravitons, we impose a covariant normal ordering prescription which has the effect of removing tadpole graphs. This version of the theory exhibits a secular slowing at three loop order due to interactions between virtual infrared scalars which are ripped apart by the inflating background. The effect is quantified using an invariant observable and all orders bounds are given. We also argue that, although stochastic effects can have either sign, the slowing mechanism is superimposed upon them.

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# 1 Introduction

The application of a force field in quantum field theory generally rearranges virtual quanta and thereby induces currents and/or stresses which modify the original force field. This is the phenomenon of *back-reaction*. Famous examples include the response of QED to a homogeneous electric field [1] and the response of generic matter theories to the gravitational field of a black hole [2]. In the former case, virtual  $e^+e^-$  pairs can acquire the energy needed to become real by tunneling up and down the field lines. The newly created pairs are also accelerated in the electric field, which gives a current that reduces the original electric field. The event horizon of a black hole also causes particle creation when one member of a virtual pair passes out of causal contact with the other by entering the event horizon. As the resultant Hawking radiation carries away the black hole's mass, the surface gravity rises.

Parker [3] was the first to realize that the expansion of spacetime can lead to the production of massless, minimally coupled scalars. Grishchuk [4] later showed that the same mechanism applies to gravitons. Production of these particles is especially efficient during inflation because virtual infrared pairs become trapped in the Hubble flow and are ripped apart from one another. Since the mechanism requires both effective masslessness on the scale of inflation and the absence of conformal invariance, it is limited to gravitons and to light, minimally coupled scalars.

Our special interest is the back-reaction from this process. We believe that the gravitational attraction between virtual infrared gravitons gradually builds up a restoring force that impedes further inflation. This mechanism offers the dazzling prospect of simultaneously resolving the (old) problem of the cosmological constant and providing a natural model of inflation in which there is no scalar inflaton [5]. The idea is that the actual cosmological constant is not small and that this is what caused inflation during the early universe. Back-reaction plays the crucial role of ending inflation.

The purpose of this paper is to establish that there is a significant back-reaction. This might appear obvious in view of the limitless extent of particle creation otherwise. It is easy to show that about one infrared pair emerges per Hubble time in each Hubble volume. Denying that there is significant back-reaction implies that an observer can watch this go on in the space around him for an arbitrarily long period without feeling any effect.

However, the preponderance of expert opinion is highly doubtful about the existence of a significant back-reaction. Some are concerned with the methodology of previous studies of back-reaction. In the pure gravity model described above, the expectation value of the gauge-fixed metric was computed in the presence of a state which is free graviton vacuum at  $t = 0$ , and the resulting invariant element was reported in co-moving coordinates,

$$\langle \Omega | g_{\mu\nu}(t, \vec{x}) dx^\mu dx^\nu | \Omega \rangle = -dt^2 + e^{2b(t)} d\vec{x} \cdot d\vec{x} . \quad (1)$$

When two loop effects are included the expansion rate is,

$$\dot{b}(t) = H \left\{ 1 - \left( \frac{G\Lambda}{3\pi} \right)^2 \left[ \frac{1}{6}(Ht)^2 + O(Ht) \right] + O(G^3) \right\} , \quad (2)$$

where  $G$  is Newton's constant,  $\Lambda$  is the cosmological constant and  $H \equiv \sqrt{\Lambda/3}$  is the Hubble constant of the locally de Sitter background [6]. Unruh has criticized the procedure of taking the expectation value of the metric first and then forming it into an invariant measure of the expansion rate [7]. Linde has no objection to gauge fixing but he believes that expectation values obscure important stochastic effects [8].

There are also concerns about the putative physical mechanism behind screening. Some doubt the causality of gravitational interactions between particles which have been pulled out of causal contact with one another. Others concede the reality of such a residual interaction but maintain that it must be redshifted into insignificance by the inflationary expansion of space-time. Finally, there are those who insist that back-reaction must degenerate to a small increase in the expansion rate because the time and space average stress-energy tensor induced by inflationary particle creation is that of a small, positive cosmological constant.

Our response to plausible methodological concerns is cooptation. We believe that a physical process such a back-reaction will manifest itself in any reasonable formalism. To address Unruh's objection we form the metric operator into an invariant measure of the local expansion rate *before* taking its expectation value [9]. To address Linde's objection we set the formalism up so that stochastic samples of this expansion operator can be taken instead of expectation values [10].

These changes are too complicated to be implement in the pure gravity model but they can be carried out in a scalar analog which possesses the

same combination of attractive self-interactions between massless, conformally noninvariant quanta. The model is essentially a massless, minimally coupled scalar with quartic self-interaction,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} - \frac{1}{4!}\lambda\phi^4\sqrt{-g} \\ + \text{counterterms} + \text{ordering corrections} , \quad (3)$$

in a locally de Sitter background. To make this system more similar to gravity we remove tadpole graphs through a procedure of covariant normal ordering that preserves conservation of the stress-energy tensor. When this is done one can apply the same formalism which was used for pure gravity to show that the expansion rate is slowed by an amount which eventually becomes non-perturbatively strong [11],

$$\dot{b}(t) = H \left\{ 1 - \frac{\lambda^2 G \Lambda}{2^7 3^4 \pi^5} \left[ (Ht)^4 + O(H^3 t^3) \right] + O(\lambda^3, G^2) \right\} . \quad (4)$$

We shall demonstrate that this result is not changed by using an invariant operator *à la* Unruh, and that it is superimposed upon an indeterminate effect of order  $\lambda$  when one takes stochastic samples *à la* Linde.

Section 2 addresses the various non-methodological objections (causality, redshift, and modeling the source as a homogeneous, classical fluid). Section 3 motivates our decision to order the scalar analog theory so as to subtract off tadpoles. Section 4 explains the actual procedure for accomplishing this. In Section 5 we show that the expectation value of the invariant expansion operator agrees exactly with (4). We also argue that, while a stochastic sample can show an effect of either sign at order  $\lambda$ , there is no significant change in the order  $\lambda^2$  result. Our conclusions comprise Section 6.

## 2 Physics of inflationary back-reaction

The purpose of this section is to review the physics of inflationary back-reaction so as to answer the three non-methodological objections which were summarized before. We shall also give a simple explanation of why back-reaction induces an ever-increasing, negative vacuum energy in perturbation theory. The aim here is not rigor — that is supplied by the detailed calculations of Section 5. We seek rather to explain in simple terms why the calculations turn out as they do.

It is worthwhile recalling that the inflationary production of gravitons (and light, minimally coupled scalars) is a straightforward consequence of the Uncertainty Principle, the existence of a causal horizon during inflation, and the simultaneous masslessness and absence of conformal invariance of the quanta being produced. The Uncertainty Principle requires all quantum degrees of freedom to possess 0-point motion. In a quantum field theory this means that virtual quanta continuously emerge from the vacuum. Although the fact of their emergence does not depend upon the background geometry, what happens to them subsequently does. In particular, if there is a causal horizon then long wave length quanta *emerge* out of contact with one another and so can never recombine. For this to happen with an appreciable amplitude the quanta must of course be massless on the scale of the horizon. They must also be sensitive to the local geometry. For the conformally flat backgrounds characteristic of long periods of inflation this implies that the quanta must not possess classical conformal invariance. That is why gravitons and light, minimally coupled scalars experience super-adiabatic amplification during inflation whereas vector gauge bosons, spin 1/2 fermions, and conformally coupled scalars do not.

In considering the gravitational back-reaction from inflationary particle production note first that there is no buildup of particle density because the 3-volume expands as new particles are created so as to keep the density constant. When a new pair is pulled out of the vacuum the one before it is, on average, already in another Hubble volume. However, the gravitational field is another thing. The created particles are highly infrared (by the standards prevailing at the time) so they do not carry very much stress-energy, but they do carry some. This must engender a gravitational field in the region between them. Because gravity is attractive, this field must act to resist the Hubble flow. This is a very small effect because gravity is a weak interaction, even on the scales usually proposed for inflation. The feature that permits it to become significant is the cumulative nature of the effect. Even after a newly created pair has been pulled into distinct Hubble volumes its gravitational field must remain behind to add with those of subsequent pairs. If nothing else supervenes to end inflation first the gravitational field must eventually become nonperturbatively strong.

There is little doubt that inflationary particle production goes on because it is the usual explanation for the primordial spectrum of cosmological perturbations [12, 13] whose imprint on the cosmic microwave background

has been so clearly imaged by the latest balloon experiments [14]. Nor is there any real doubt that *some* gravitational back-reaction accrues from the process, since the usual theory of structure formation holds that we live in complex structures resulting from the gravitational collapse of primordial fluctuations over the course of ten billion years. The real issues are, whether or not there is a gravitational response *during* inflation, whether or not this response grows with time, and whether or not the response slows inflation.

Those who argue against any response at all base themselves on causality. The source for our process is virtual particles which rapidly fall out of causal contact with one another. Therefore, how can they interact in *any* way? The flaw in this argument is the picture it implies of exchange forces being maintained by instantaneous interaction across some arbitrary surface of simultaneity. Were this correct an outside observer could not feel the gravitational attraction from a pebble after it had entered the event horizon (in some arbitrary frame) of a black hole. In fact exchange forces are maintained by virtual quanta which propagate causally, and those which carry the gravitational force from an object falling into a black hole originate in the region *outside* the event horizon. The same is true of objects which fall out of causal contact in an inflating universe: they continue to feel a gravitational force from one another that is carried in a completely causal manner from far back along their past light cones. The process is depicted in Fig. 1.

Three other points deserve mention in connection with the issue of causality. The first is that inflationary particle production would not occur at all but for the causal horizon of an inflating universe. Far from being something we ignore, causality is at the heart of the effect. The second point is that the explicit perturbative computations [6, 11] which support this picture were made using the Schwinger-Keldysh formalism [15]. This method is *manifestly* causal: interaction vertices from outside the past lightcone of the observation point make no contribution. The final point is that the effects of causality are quite evident in the factors of  $Ht$  which appear in the result (2). They are present, in essence, because two interaction vertices are being integrated over the volume of the past light cone (see Fig. 2.), each factor of which grows like  $Ht$ .

Those who argue against the possibility of a secular response concede that particles continue to attract one another after having fallen out of causal contact. But they maintain that the lines of force from this effect must be rapidly be diluted owing to the inflationary expansion of spacetime. Therefore the

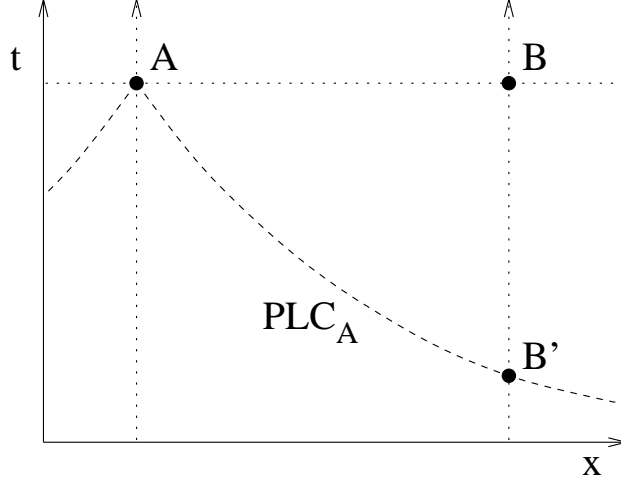


Figure 1: The worldlines of two freely falling particles  $A$  and  $B$  in co-moving coordinates. The interaction  $A$  feels from  $B$  derives not from some arbitrary surface of simultaneity but rather from the portion of its past light cone ( $B'$  and below) in which  $B$  is visible.

gravitational fields contributed from particles created early during inflation should become weaker and weaker. Were this view correct the cumulative gravitational potential would not be the integral of a constant — as we have argued — but rather some power of the ratio of past to present scale factors,

$$\int_0^t dt' e^{b(t')-b(t)} \sim \frac{1}{\dot{b}(t)}, \quad (5)$$

Note that we are assuming a homogeneous, isotropic and spatially flat background geometry,

$$ds^2 = -dt^2 + e^{2b(t)} d\vec{x} \cdot d\vec{x} = \Omega^2(\eta)[d\eta^2 + d\vec{x} \cdot d\vec{x}]. \quad (6)$$

We have also used the slow roll approximation to evaluate the integral.<sup>1</sup>

There is no question that the argument given above applies for a conformally invariant force field such as electromagnetism. But gravity is not conformally invariant, and the gravitational response from early perturbations approaches a constant rather than redshifting to zero [16]. Because it

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<sup>1</sup>This means  $\dot{b}^N \gg |b^{(N)}|$  and also that  $e^{b(t)} \gg 1$  for  $Ht \gg 1$ .

illustrates the crucial distinction between gravitation and electromagnetism we will make the argument in detail. Consider an electromagnetic analog of the particle creation process in which a homogeneous and temporally constant current density  $J^i(t, \vec{x}) = j\delta^i_3$  is produced at each point in an inflating spacetime. In flat space this would engender a linearly growing electric field of the form,  $E^i(t, \vec{x}) = -jt\delta^i_3$ , whose interpretation would be the superposition over time of a constant electric displacement.

Things work a little differently during inflation. The curved space Maxwell equations are,

$$\partial_\nu (\sqrt{-g}F^{\nu\mu}) = J^\mu \sqrt{-g} . \quad (7)$$

For our problem only the  $\mu = 3$  component is nontrivial,

$$-\frac{d}{dt} (e^{3b(t)} F^{30}(t)) = j e^{3b(t)} . \quad (8)$$

If we assume zero initial electric field the solution is,

$$E^3(t) \equiv F^{30}(t) = -j \int_0^t dt' (e^{b(t')-b(t)})^3 \sim \frac{-j}{3\dot{b}(t)} . \quad (9)$$

The continuous current produces an electric field which approaches a constant during inflation (when  $\dot{b}$  is nearly constant). There is no appreciable buildup from previous times because the electric field lines redshift like  $e^{-2b}$ .<sup>2</sup>

To see what happens without conformal invariance let us first translate the electrodynamic problem to the context of a massless, conformally coupled scalar. The Lagrangian is,

$$\mathcal{L}_{\text{MCC}} = -\frac{1}{2}\partial_\mu\psi\partial_\nu\psi g^{\mu\nu}\sqrt{-g} - \frac{1}{12}\psi^2 R\sqrt{-g} - J\psi\sqrt{-g} . \quad (10)$$

Specializing its field equation to a constant source  $J(x) = j$  in a homogeneous and isotropic geometry gives,

$$\frac{1}{\sqrt{-g}}\partial_\mu (\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) - \frac{1}{6}R\psi = J \longrightarrow -e^{-2b}\frac{d}{dt} \left( e^b \frac{d}{dt} e^b \psi \right) = j . \quad (11)$$

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<sup>2</sup>This exercise incidentally illustrates the way in which sources can produce effects even when they are not within one Hubble length on some arbitrary surface of simultaneity. To see this simply turn the current off at a certain time and watch the subsequent evolution of the electric field. Although it decays exponentially it is still nonzero, despite there being no current anywhere on the same surface of simultaneity.



Assuming no initial field the solution is,

$$\psi(t) = -j e^{-b(t)} \int_0^t dt' e^{-b(t')} \int_0^{t'} dt'' e^{2b(t'')} \sim \frac{-j}{2\dot{b}^2(t)}. \quad (12)$$

Just as in the analogous electrodynamic problem, the conformally coupled scalar approaches a constant during inflation because previous contributions to it are redshifted.

We can understand what happens when conformal invariance is absent by comparing the response to the same source from a massless, *minimally* coupled scalar. Its Lagrangian is,

$$\mathcal{L}_{\text{MMC}} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - J \phi \sqrt{-g}. \quad (13)$$

The same specializations as before reduce the equation of motion to,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = J \longrightarrow -e^{-3b} \frac{d}{dt} (e^{3b} \dot{\phi}) = j. \quad (14)$$

The solution can again be expressed as a double integral, but now there is no outer redshift factor,

$$\phi(t) = -j \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} \sim -j \int_0^t dt' \frac{1}{\dot{b}(t')}. \quad (15)$$

During inflation  $\dot{b}$  is nearly constant, so  $\phi(t) \sim -jt/\dot{b}$ . Hence we learn that lifting conformal invariance permits early sources to contribute on an equal footing with late ones.

The scalar comparison we have just given was far from specious. It has long been known that, dynamical gravitons in a homogeneous and isotropic background geometry obey the same linearized equation as massless, minimally coupled scalars [4]. It is also well known that a single perturbation of the metric does not redshift away during inflation. Instead the (synchronous gauge) perturbed metric approaches the form [16, 17],

$$g_{\mu\nu}(t, \vec{x}) dx^\mu dx^\nu \longrightarrow -dt^2 + e^{2b(t)} \gamma_{ij}(\vec{x}) dx^i dx^j. \quad (16)$$

Boucher and Gibbons prove their “cosmic baldness” theorem by then arguing that, since the temporally constant, spatial variation of  $\gamma_{ij}(\vec{x})$  must fall off

above a certain co-moving Fourier mode, any freely falling local observer is eventually unable to sense it. This is true for *one* perturbation, but what happens when there is a mechanism, such as the Uncertainty Principle, which generates perturbations with higher and higher co-moving wave numbers? There is simply no escaping the fact that the amplitude of  $\gamma_{ij}$  will change more and more, and that this change will be visible to local observers. Hence the effect is not only real but also secular.

This brings us to the third objection of principle, that inflationary particle production should result in a small fractional *increase* in the expansion rate. The basis of this argument is modeling the stress-energy of particle production as a homogeneous and isotropic classical fluid. If this were correct we could compute the gravitational back-reaction using the homogeneous and isotropic Einstein equations,

$$3\dot{b}^2(t) = 3H^2 + 8\pi G\rho(t) , \quad (17)$$

$$-2\ddot{b}(t) - 3\dot{b}^2(t) = -3H^2 + 8\pi Gp(t) . \quad (18)$$

It is straightforward to show that locally de Sitter inflation at Hubble constant  $\mathcal{H}$  results in an *average* energy density and pressure of  $\rho = -p = \mathcal{H}^4/16\pi^2$  per species of massless, minimally coupled scalar.<sup>3</sup> Substituting  $\dot{b} = \mathcal{H}$  and solving the resulting quadratic equation gives the following result for the final expansion rate,

$$\mathcal{H}^2 = \frac{3\pi}{2G} \left( 1 - \sqrt{1 - \frac{4G}{3\pi} H^2} \right) = H^2 \left\{ 1 + \frac{G\Lambda}{9\pi} + 2 \left( \frac{G\Lambda}{9\pi} \right)^2 + \dots \right\} . \quad (19)$$

That something must be wrong with this argument becomes apparent from the evident fact that it requires the expansion rate to increase. In other words, before the steady state solution (19) is attained we must have  $\ddot{b} = -4\pi G(\rho + p) > 0$ . But this violates the weak energy condition! Such a thing *can* happen through quantum effects [19], but the physics is hopelessly wrong in this case. A collection of particles, even gravitons, should be a drag on the expansion of spacetime, not a super-accelerant to it.

Closer inspection reveals two flaws in the argument. First, it assumes that the distribution of created particles is homogeneous and isotropic. This can hardly be so when the density of these quanta is about one per Hubble

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<sup>3</sup>For a detailed derivation of this old and well-known result, see Section 4 of [18].

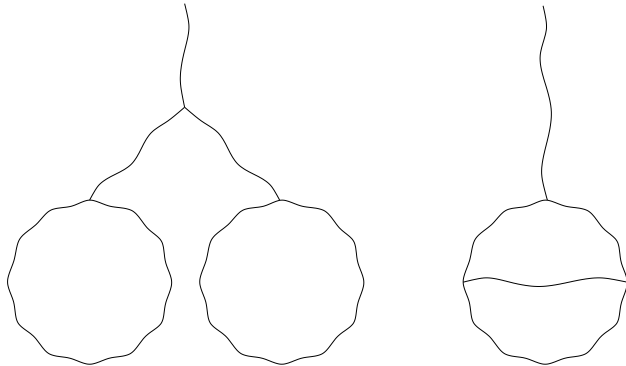


Figure 2: Two loop contributions to the background geometry from gravitons. The lefthand diagram is a negligible constant because the gravitons do not interact in a correlated fashion. The righthand one slows expansion by a fractional amount which grows like  $(G\Lambda)^2(Ht)^2$  because the gravitons are correlated when they interact.

volume. Gravitons move at the speed of light, and there must necessarily be a *direction* associated with this motion. The one infrared graviton in any single Hubble volume cannot give rise to a stress-energy tensor which is even approximately isotropic. Pretending otherwise risks the same sort of mistake as using the zero average charge density to describe the dielectric properties of a ponderable medium which actually consists of an enormous number of positive and negative charges.

The second flaw is ignoring quantum correlations. Fig. 2 presents two diagrams that contribute to the quantum gravitational back-reaction on an inflating universe. Each has two loops and contributes at order  $(G\Lambda)^2$ . Yet the diagram on the left gives a constant that can be absorbed into renormalizing the cosmological constant<sup>4</sup> whereas the one on the right contributes to the secular slowing of the Hubble constant in expression (2) [6]. The key distinction between the two graphs is that the virtual gravitons on the right are correlated when they interact whereas the ones on the left are not.

It is easy to understand why correlation can make so much difference. Consider the traveling wave pulses depicted in Fig. 3. If the stretched string upon which they move is exactly linear then the packets pass through one

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<sup>4</sup>In fact it *must* be so absorbed in order for the initial condition of inflation with Hubble constant  $H$  to be satisfied.

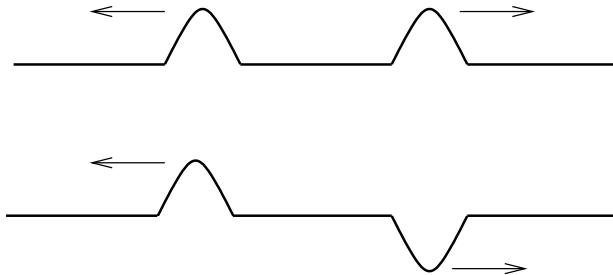


Figure 3: A simple illustration of how correlation can change an interaction. The wave packets on the top have the same sign, so were there a fourth order coupling they would repel one another. The wave packets on the bottom have opposite signs. With the same coupling they attract one another.

another with no effect, however we can imagine that there is a nonlinear interaction whose contribution to the potential energy of the string goes like the fourth power of the displacement. Then it is clear that pulses of the same sign and amplitude  $A$  repel one another because the superposed amplitude of  $2A$  gives a potential energy of  $+16A^4$ , which is larger than the  $2A^4$  when they are far apart. On the other hand, pulses with the opposite sign attract because their amplitude together is zero, which means zero potential as compared with the same  $2A^4$  when far apart. Pulses of the opposite sign are a reasonable way (with classical string!) of representing particle-anti-particle pairs. Ignoring their affinity for one another — which is what using the average stress-energy does — is a terrible approximation. It doesn't even get the sign right!

In considering the effects of anisotropies and correlations it is important to distinguish between the bare stress-energy of created gravitons — just the  $\hbar\omega$  per particle — and the stress-energy they develop through gravitational interaction with one another. The former is highly anisotropic and also highly correlated, which is why one doesn't even get the right sign by using its average value to compute the gravitational interaction energy. What the various elements of such a source must actually do is to attract one another, whereas a precisely homogeneous and isotropic source causes spacetime to expand. However, the gravitational interaction at any point is the result of superposing the gravitational fields of all pairs that have been created in the past light cone of that point. There are no significant quantum correlations

between different pairs, and their individual anisotropies tend to average out over a long period of inflation. So it should be valid to infer the effect on inflation by using (17-18), provided it is the energy density and pressure of gravitational interaction that are substituted, and provided that these quantities are not themselves computed by using the average of the bare stress-energy in the homogeneous and isotropic Einstein equations.

Of course the correct way of inferring the stress-energy of gravitational interaction is from the operator equations of quantum general relativity, and precisely that was done to obtain (2) [6]. However, we can understand the result by simply forcing the average bare energy density of created particles to attract itself as we know that the actual, anisotropic and quantum correlated source does. Much of the essential physics can even be captured using Newtonian gravity to estimate the interaction energy, provided the pressure is assumed to follow from conservation.

Consider locally de Sitter inflation on a manifold whose spatial section is a 3-torus. If the physical radius of the universe is initially  $H^{-1}$  then its value at co-moving time  $t$  is,

$$r(t) \sim H^{-1} e^{Ht} . \quad (20)$$

As mentioned before, the average of the bare energy density of inflationally produced infrared gravitons is,

$$\rho_{\text{IR}} \sim H^4 . \quad (21)$$

This is insignificant compared with the energy density in the cosmological constant ( $\sim H^2/G$ ), and  $\rho_{\text{IR}}$  is in any case positive. However, the gravitational interaction energy is negative, and it can be enormous if there is contact between a large enough fraction of the total mass of infrared gravitons,

$$M(t) \sim r^3(t) \rho_{\text{IR}} \sim H e^{3Ht} . \quad (22)$$

For example, if  $M(t)$  was *all* in contact with itself the Newtonian interaction energy would be,

$$- \frac{GM^2(t)}{r(t)} \sim -GH^3 e^{5Ht} , \quad (23)$$

Dividing by the 3-volume gives a density of about  $-GH^6 e^{2Ht}$ , which rapidly becomes enormous.

Of course this ignores causality. Most of the infrared gravitons needed to maintain  $\rho_{\text{IR}}$  are produced out of causal contact with one another in different Hubble volumes. The ones in gravitational interaction are those produced within the same Hubble volume. Since the number of Hubble volumes grows like  $e^{3Ht}$ , the rate at which mass is produced within a single Hubble volume is,

$$\frac{dM_1}{dt} \sim H^2 . \quad (24)$$

Although most of the newly produced gravitons soon leave the Hubble volume, their gravitational potentials must remain, just as an outside observer continues to feel the gravity of particles that fall into a black hole. The rate at which the Newtonian potential accumulates is therefore,

$$\frac{d\Phi_1}{dt} \sim -\frac{G}{H^{-1}} \frac{dM_1}{dt} \sim -GH^3 . \quad (25)$$

Hence the Newtonian gravitational interaction energy density is,

$$\rho(t) \sim \rho_{\text{IR}} \Phi_1(t) \sim -GH^6 Ht \sim -(GH^2)^2 \cdot Ht \cdot \frac{H^2}{G} . \quad (26)$$

Although this model is very crude it does act in the right sense — to slow inflation — and it is secular. The dependence upon coupling constants is also correct —  $(G\Lambda)^2$ , characteristic of a 2-loop process like the quantum gravitational result (2). This can be understood from the fact that one must go to one loop order to see particle production, whereas the interactions between these particles — which is the source of the effect — requires another order in perturbation theory.

One other thing that this model gets right is the equation of state, if we also assume the relativistic form of stress-energy conservation. When the number of e-foldings  $Ht$  becomes large, the fractional rate of change of the gravitational interaction energy is negligible compared with the expansion rate,

$$|\dot{\rho}(t)| \ll H|\rho(t)| . \quad (27)$$

It follows from energy conservation,

$$\dot{\rho}(t) = -3H(\rho(t) + p(t)) , \quad (28)$$

that the induced pressure must be nearly opposite to the energy density. In other words, back-reaction induces negative vacuum energy. Since the key requirement is slow accumulation in the sense of relation (27), the equation of state is really a consequence of the fact that gravity is a weak interaction on the scale of inflation.

Before closing the section we should also comment that one does not require a complete solution of quantum gravity in order to study an infrared process such as this. As long as spurious time dependence is not injected through the ultraviolet regularization, the late time back-reaction is dominated by ultraviolet finite, nonlocal terms whose form is entirely controlled by the low energy limiting theory. This theory must be general relativity,

$$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)\sqrt{-g} , \quad (29)$$

with the possible addition of some light scalars. Here “light” means massless with respect to  $H \equiv \sqrt{\Lambda/3}$ . No other quanta can contribute effectively in this regime.

It is worth commenting that infrared phenomena can always be studied using the low energy effective theory. This is why Bloch and Nordsieck [20] were able to resolve the infrared problem of QED before the theory’s renormalizability was suspected. It is also why Weinberg [21] was able to achieve a similar resolution for  $\Lambda = 0$  quantum general relativity. And it is why Feinberg and Sucher [22] were able to compute the long range force due to neutrino exchange using Fermi theory. More recently Donoghue [23] has been working along the same lines for  $\Lambda = 0$  quantum gravity.

### 3 Problems with coincident propagators

As discussed in the previous section, the effect we seek to study has two essential features. The first is that infrared virtual particles are continually being ripped out of the vacuum and pulled apart by the inflationary expansion of spacetime. The second crucial feature is that these particles attract one another through a weak long range force which gradually accumulates as more and more particles are created. The particles we believe were actually responsible for stopping inflation are gravitons, and the long range force through which they did it was gravitation. However, this is not a simple theoretical setting in which to work. It took over a year of labor to obtain the

two loop result (2) — even with computer symbolic manipulation programs [6]. Before attempting to extend this feat to include invariant observables or stochastic samples one naturally wonders whether there is not some simpler theory we could study which manifests the same effect.

As also explained in the previous section, the prerequisites for inflationary particle production are masslessness on the scale of inflation and the absence of classical conformal invariance. Massless, minimally coupled scalars have these properties — and the lowest order back-reaction from self-interacting scalars can be worked out on a blackboard in about 15 minutes [11]. Of course it is not natural for scalars to possess attractive self-interactions and still remain massless on the scale of inflation. But we do not need a *realistic* model — that is already provided by gravitation. What we seek is rather a *simple* model that can be tuned to show the same physics, however contrived and unnatural this tuning may be.

The behavior of free, massless and minimally coupled scalars in a locally de Sitter background has been much studied [24, 25, 26, 27]. Among the curious properties of these particles are the absence of any normalizable, de Sitter invariant states [24] and the appearance of acausal infrared singularities when the Bunch-Davies vacuum is used with infinite 3-surfaces [25, 26]. The feature that concerns us here is the assertion that taking the coincidence limit of the propagator gives an ultraviolet divergent constant plus a finite term which grows linearly with the co-moving time [27],

$$i\Delta(x; x) = UV + \frac{H^2}{4\pi^2} Ht . \quad (30)$$

Although this may not be an observable statement for free scalars we shall argue in this section that it has three disturbing consequences for our program of adding a  $\lambda\phi^4$  self-interaction to obtain a paradigm for infrared quantum gravity:

1. The linear growth derives from the ultraviolet, not the infrared;
2. Scalars develop a linearly growing mass at order  $\lambda$ ; and, worst,
3. The stress-energy violates the weak energy condition at order  $\lambda$ .

The purpose of this section is to demonstrate these problems. We will explain how to correct them in the next section.



We begin by giving the Lagrangian. Without ordering corrections it is,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} - \frac{1}{4!}\lambda\phi^4\sqrt{-g} + \Delta\mathcal{L} . \quad (31)$$

The various counterterms reside in  $\Delta\mathcal{L}$ ,

$$\begin{aligned} \Delta\mathcal{L} = & -\frac{1}{2}\delta m^2\phi^2\sqrt{-g} - \frac{1}{12}\delta\xi(R-4\Lambda)\phi^2\sqrt{-g} - \frac{\delta\Lambda}{8\pi G}\sqrt{-g} , \\ & -\frac{1}{2}\delta Z\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} - \frac{1}{4!}\delta\lambda\phi^4\sqrt{-g} . \end{aligned} \quad (32)$$

The ones on the first line are of order  $\lambda$  and will figure in the considerations of this section. The ones on the second line are of order  $\lambda^2$  and will not concern us further here.

We are not quantizing gravity. The metric is a non-dynamical background which we take to be locally de Sitter in conformal coordinates,

$$g_{\mu\nu}(\eta, \vec{x}) = \Omega^2(\eta)\eta_{\mu\nu} \quad , \quad \Omega(\eta) = -\frac{1}{H\eta} = e^{Ht} . \quad (33)$$

(Recall that  $\Lambda = 3H^2$ .) To regulate the infrared problem on the initial value surface we work on the manifold  $T^3 \times R$ , with the spatial coordinates in the finite range,  $-H^{-1}/2 < x^i \leq H^{-1}/2$ . We release the state in Bunch-Davies vacuum at  $t = 0$ , corresponding to conformal time  $\eta = -H^{-1}$ . Note that the infinite future corresponds to  $\eta \rightarrow 0^-$ , so the possible variation of conformal coordinates in either space or time is at most  $\Delta x = \Delta\eta = H^{-1}$ .

Because the spatial manifold is compact, wave numbers have the form,  $\vec{k} = 2\pi H\vec{n}$ , where  $\vec{n}$  is a vector of integers. Excepting for some completely irrelevant zero modes, the free field mode sum has the form [28],

$$\phi_I(\eta, \vec{x}) = H^3 \sum_{\vec{k}} \left\{ u(\eta, k) e^{i\vec{k}\cdot\vec{x}} a(\vec{k}) + u^*(\eta, k) e^{-i\vec{k}\cdot\vec{x}} a^\dagger(\vec{k}) \right\} , \quad (34)$$

where the properly normalized Bunch-Davies mode functions are,

$$u(\eta, k) \equiv \frac{1}{\sqrt{2k}} \left( \Omega^{-1} + \frac{iH}{k} \right) e^{-ik\eta} . \quad (35)$$

If we define  $\Delta\eta \equiv \eta - \eta'$  the product of the mode function and its conjugate can be reduced to the following useful form,

$$u(\eta, k)u^*(\eta', k) = \frac{e^{-ik\Delta\eta}}{2k\Omega(\eta)\Omega(\eta')} + \frac{H^2}{2k^3}[1 + ik\Delta\eta]e^{-ik\Delta\eta} . \quad (36)$$

This is of course the combination which occurs in the propagator.

We introduce a convergence factor of  $e^{-\epsilon k}$  to promote the mode sum for the free propagator from a distribution into a well-defined function,

$$i\Delta(x; x') = \frac{H^3}{2\Omega(\eta)\Omega(\eta')} \sum_{\vec{k}} \frac{e^{-\epsilon k}}{k} e^{-ik|\Delta\eta| + i\vec{k}\cdot\Delta\vec{x}} + \frac{H^5}{2} \sum_{\vec{k}} \frac{e^{-\epsilon k}}{k^3} [1 + ik|\Delta\eta|] e^{-ik|\Delta\eta| + i\vec{k}\cdot\Delta\vec{x}} . \quad (37)$$

The spatial separation vector and its norm are  $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$  and  $\Delta x \equiv \|\Delta\vec{x}\|$ . Because the range of conformal coordinates is rather small, it is an excellent approximation to represent the mode sum (37) as an integral. When this is done the result is amazingly simple [28],

$$i\Delta(x; x') = \frac{1}{4\pi^2} \frac{\Omega^{-1}(\eta)\Omega^{-1}(\eta')}{\Delta x^2 - (|\Delta\eta| - i\epsilon)^2} - \frac{H^2}{8\pi^2} \ln \left[ H^2 \left( \Delta x^2 - (|\Delta\eta| - i\epsilon)^2 \right) \right] + O(\Delta\eta, \Delta x) . \quad (38)$$

Note that the  $i\epsilon$  term serves as an ultraviolet regulator. It is in fact an exponential cutoff on the co-moving momentum.

We can now demonstrate the first problem by taking the coincidence limit. Setting  $\Delta\eta = \Delta x = 0$  in (38) gives,

$$i\Delta(x; x) = \frac{1}{4\pi^2} \frac{1}{\Omega^2 \epsilon^2} - \frac{H^2}{8\pi^2} \ln \left[ H^2 \epsilon^2 \right] . \quad (39)$$

This is the result with a cutoff on the co-moving momentum. For an invariant regularization we should really cut off on the *physical* momentum,  $k_{\text{phys}} \equiv \Omega^{-1}k$ . The change is easily made with the replacement  $\epsilon \rightarrow e(H\Omega)^{-1}$ ,

$$i\Delta(x; x) = \left( \frac{H}{2\pi} \right)^2 \left\{ \frac{1}{e^2} - \ln(e) + \ln(\Omega) \right\} , \quad (40)$$

$$\equiv \text{UV} + \left( \frac{H}{2\pi} \right)^2 Ht . \quad (41)$$

We are not saying this time dependence is wrong, but its origin is very clear. It comes from a logarithmic ultraviolet divergence which is being cut off at

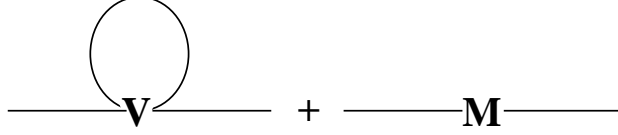


Figure 4: The scalar self-mass at order  $\lambda$  *without* normal ordering.  $V$  denotes the 4-point vertex and  $M$  stands for the mass counterterm vertex.

a time dependent point. The terrifically undesirable feature about this is that the result can change when one changes the ultraviolet structure of the theory, which is precisely the sector of quantum gravity we do not know.

To see the second problem we compute the one loop scalar self mass-squared. The diagrams are depicted in Fig. 4. A trivial application of the Feynman rules gives,

$$-iM_{1\text{-loop}}^2 = -\frac{i}{2}\lambda i\Delta(x;x) - i\delta m^2 . \quad (42)$$

The ultraviolet divergence must be absorbed with the counterterm, and the finite part is fixed by demanding that the scalar be initially massless. Therefore the mass counterterm is,

$$\delta m^2 = -\frac{\lambda}{2} \text{UV} + O(\lambda^2) , \quad (43)$$

and the renormalized one loop mass squared is,

$$M_{1\text{-loop}}^2 = \frac{\lambda}{8\pi^2} H^2 H t . \quad (44)$$

This establishes that coincident propagators lead to a linearly increasing mass at order  $\lambda$ . Of course this is undesirable because gravitons *never* develop a mass.

To see the third and worst problem, consider the order  $\lambda$  (two loop) corrections to the expectation value of the scalar stress-energy tensor. They are given in Figures 5-7. The dominant contribution comes from the diagrams in Fig. 5. Applying the Feynman rules and substituting for the mass renormalization (43) gives,

$$T_{\mu\nu}^{\text{Fig. 5}} = g_{\mu\nu} \left\{ -\frac{\lambda}{8} [i\Delta(x;x)]^2 - \frac{\delta m^2}{2} i\Delta(x;x) \right\} , \quad (45)$$

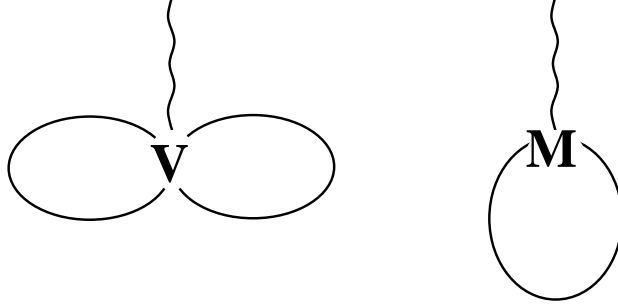


Figure 5: The dominant contributions to the scalar stress-energy tensor at order  $\lambda$  *without* normal ordering.  $V$  denotes the 4-point vertex and  $M$  stands for the mass counterterm vertex.

$$= g_{\mu\nu} \left\{ +\frac{\lambda}{8} UV^2 - \left( \frac{H}{2\pi} \right)^4 (Ht)^2 \right\} . \quad (46)$$

Note the bizarre nature of the stress-energy tensor with this term. Inflation actually speeds up. In fact there is a trivial interpretation for what is happening: the Uncertainty Principle causes the scalar to wander from its classical value of  $\phi = 0$ , and that engenders a positive potential energy. It is this scalar potential energy that increases the expansion rate.

Of course the diagrams of Fig. 5 are not the only ones which contribute to the stress-energy tensor. Although the others cannot change the  $(Ht)^2$  terms, they do add important structure to enforce stress-energy conservation and cancel the ultraviolet divergences.<sup>5</sup> Hence our leading order results for the induced energy density ( $T_{00} \equiv -\rho g_{00}$ ) and pressure ( $T_{ij} \equiv pg_{ij}$ ),

$$\rho = \frac{\lambda}{8} \left( \frac{H}{2\pi} \right)^4 \left\{ (Ht)^2 + O(Ht) \right\} + O(\lambda^2) , \quad (47)$$

$$p = -\frac{\lambda}{8} \left( \frac{H}{2\pi} \right)^4 \left\{ (Ht)^2 + O(Ht) \right\} + O(\lambda^2) , \quad (48)$$

---

<sup>5</sup>Of particular interest is the conformal counterterm given in the first diagram of Fig. 7. In a locally de Sitter background there is no distinction between  $R$  and the constant,  $12H^2$ . However, the distinction *does* matter in computing the stress-energy tensor and one must use the conformal counterterm to cancel overlapping divergences which come from the diagrams of Fig. 6.

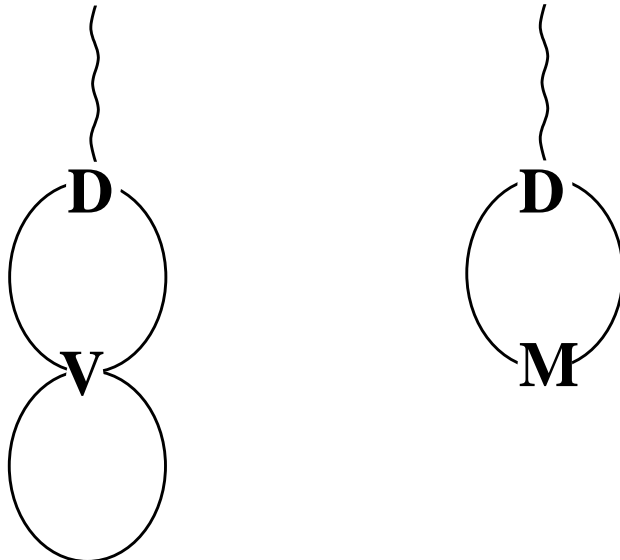


Figure 6: Contributions to the scalar stress-energy tensor at order  $\lambda$  which involve the derivative vertex  $D$  *without* normal ordering.  $V$  denotes the 4-point vertex and  $M$  stands for the mass counterterm vertex.

imply that their sum is,

$$\rho + p = -\frac{\dot{\rho}}{3H} = -\frac{\lambda}{8} \left( \frac{H}{2\pi} \right)^4 \left\{ \frac{2}{3} Ht + O(1) \right\} + O(\lambda^2) . \quad (49)$$

Note that  $\rho + p$  is not changed if we include the energy density and pressure of the bare cosmological constant through the replacements,  $\rho \longrightarrow \Lambda/8\pi G + \rho$  and  $p \longrightarrow -\Lambda/8\pi G + p$ . This establishes the third and final problem with coincident propagators: they result in a violation of the weak energy condition at order  $\lambda$ .

## 4 Covariant normal ordering

We would like to have a theory in which the ultraviolet does not contaminate the infrared, in which massless particles remain massless, and in which matter is a drag on expansion rather than a super-accelerant. The purpose of this section is to describe a consistent way in which the scalar model can be

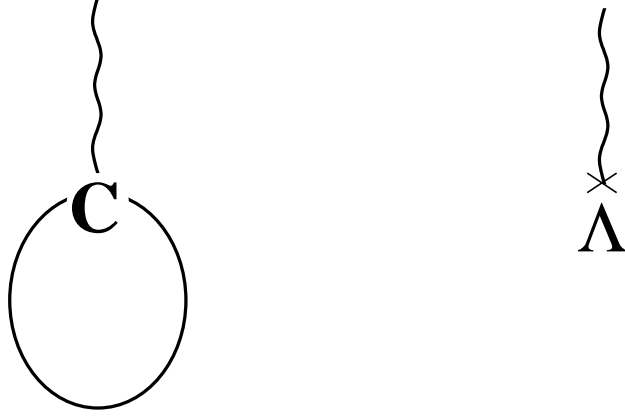


Figure 7: Contributions to the scalar stress-energy tensor at order  $\lambda$  from the conformal counterterm  $C$  and the counterterm for the bare cosmological constant  $\Lambda$ .

altered to remove these undesirable features, at least at the lowest orders in  $\lambda$ . We call the method, *covariant normal ordering*. For simplicity we shall give the result for the original Lagrangian, without its counterterms. Their inclusion is straightforward.

In what follows we assume that the theory has been regulated invariantly so that the coincident propagator is a finite scalar functional of the metric. We define the normal-ordered product of  $\phi^N$  as follows,

$$:\phi^N(x): \equiv \sum_{k=0}^{[N/2]} \frac{(2k-1)!!N!}{(2k)!(N-2k)!} (-i\Delta(x;x))^k \phi^{N-2k}(x). \quad (50)$$

Note that one can differentiate either with respect to the field or the coincident propagator,

$$\frac{\partial : \phi^N :}{\partial \phi} = N : \phi^{N-1} : \quad , \quad \frac{\partial : \phi^N :}{\partial (-i\Delta)} = \frac{N(N-1)}{2} : \phi^{N-2} : \quad . \quad (51)$$

The trick behind covariant normal-ordering is to implement it at the level of the Lagrangian through the replacement:  $\mathcal{L} \longrightarrow : \mathcal{L} :$ . In this way all objects derived from the action — such as the scalar stress-energy tensor and the scalar field equations — are free of tadpoles.

Stress-energy conservation is maintained by taking account of the implicit metric dependence of the propagator. One can infer this, formally, from the functional integral representation,

$$i\Delta(y; y') = \left[ d\phi \right] \phi(y) \phi(y') \exp \left[ -\frac{i}{2} \int d^4x \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} \right] . \quad (52)$$

Hence the variation which gives the stress-energy tensor produces,

$$\begin{aligned} \frac{-2}{\sqrt{-g(x)}} \frac{\delta i\Delta(y; y')}{\delta g^{\mu\nu}(x)} &= i \left[ \delta^\alpha_{(\mu} \delta^\beta_{\nu)} - \frac{1}{2} g_{\mu\nu}(x) g^{\alpha\beta}(x) \right] \\ &\times \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} \{ i\Delta(x; x') i\Delta(y; y') + 2i\Delta(x; y) i\Delta(x'; y') \}_{x'=x} . \end{aligned} \quad (53)$$

However, we shall modify this scheme in two ways, one necessary and the other highly convenient. The convenient modification is that we can drop the first of the three terms in (53) because it is separately conserved.

The necessary modification is that one really varies the Schwinger functional integral [29] to obtain the stress-energy tensor. This contains  $+$  fields which evolve the theory forward and  $-$  fields which evolve it back to the initial state. Although there is no mixing between these fields, *both* are minimally coupled to the same metric. Hence the stress-energy tensor receives contributions from both terms. This is necessary to make the stress-energy tensor real and to make it depend causally upon quantities in the past light-cone of the point  $x^\mu$  at which it is evaluated. Two sorts of propagators result,

$$\begin{aligned} i\Delta_{++}(x; x') &\equiv \\ &\frac{1}{4\pi^2} \frac{\Omega^{-1}(\eta) \Omega^{-1}(\eta')}{\Delta x^2 - (|\Delta\eta| - i\epsilon)^2} - \frac{H^2}{8\pi^2} \ln \left[ H^2 \left( \Delta x^2 - (|\Delta\eta| - i\epsilon)^2 \right) \right] , \end{aligned} \quad (54)$$

$$\begin{aligned} i\Delta_{+-}(x; x') &\equiv \\ &\frac{1}{4\pi^2} \frac{\Omega^{-1}(\eta) \Omega^{-1}(\eta')}{\Delta x^2 - (\Delta\eta + i\epsilon)^2} - \frac{H^2}{8\pi^2} \ln \left[ H^2 \left( \Delta x^2 - (\Delta\eta + i\epsilon)^2 \right) \right] . \end{aligned} \quad (55)$$

Of course the  $++$  propagator is just the same as the Feynman one,  $i\Delta(x; x')$ . Note also that whereas the free kinetic operator acts upon  $i\Delta_{++}(x; x')$  to give a delta function, it annihilates  $i\Delta_{+-}(x; x')$ .

With the two modifications described above the result is,

$$\begin{aligned}
T_{\mu\nu}(x) = & \left[ \delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \right] : \partial_\rho \phi(x) \partial_\sigma \phi(x) : - g_{\mu\nu}(x) \frac{\lambda}{4!} : \phi^4(x) : \\
& + \frac{i\lambda}{2} \left[ \delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \right] \int d^4 y \sqrt{-g(y)} : \phi^2(y) : \theta(y^0 + H^{-1}) \\
& \times [\partial_\rho i\Delta_{++}(x; y) \partial_\sigma i\Delta_{++}(x; y) - \partial_\rho i\Delta_{+-}(x; y) \partial_\sigma i\Delta_{+-}(x; y)] . \quad (56)
\end{aligned}$$

Note that  $++$  and  $+-$  propagators interfere destructively whenever the dummy variable of integration,  $y^\mu$ , strays outside the past lightcone of the observation point  $x^\mu$ . Using the normal-ordered field equations,

$$\frac{\delta : S :}{\delta \phi(x)} = \partial_\mu \sqrt{-g(x)} g^{\mu\nu}(x) \partial_\nu \phi(x) - \frac{\lambda}{6} : \phi^3(x) : \sqrt{-g(x)} = 0 , \quad (57)$$

it is easy to see that the stress-energy tensor is conserved.

Covariant normal-ordering is trivial to use: simply apply the old Feynman rules and then ignore any coincident propagators. Including the counterterms is straightforward but irrelevant because the first contributions to  $\delta m^2$ ,  $\delta \xi$ ,  $\delta Z$  and  $\delta \lambda$  are of order  $\lambda^2$ . Since the additional terms in the stress-energy operator consist of these  $O(\lambda^2)$  constants times normal-ordered products of the fields, the lowest correction to the stress-energy tensor from all except the  $\delta \Lambda$  counterterm are of order  $\lambda^3$ . This cannot affect the order  $\lambda^2$  effect we shall compute in the next section.

Covariant normal-ordering cannot be fundamental because it results in a stress-energy tensor that depends nonlocally (but causally) upon the fields. However, it does seem to be acceptable if all we seek is a method for tuning a simple, scalar model to make it roughly agree with what goes on in the vastly more complicated system of quantum general relativity. In particular, the model's stability should be enhanced, not endangered, by eliminating the tendency of quantum fluctuations to produce super-acceleration at the lowest order in perturbation theory. It is also relevant to note that the technique reduces to ordinary normal-ordering in the flat space limit.

## 5 Back-reaction

The purpose of this section is to demonstrate that the scalar model defined by covariant normal-ordering shows real back-reaction at the lowest order in



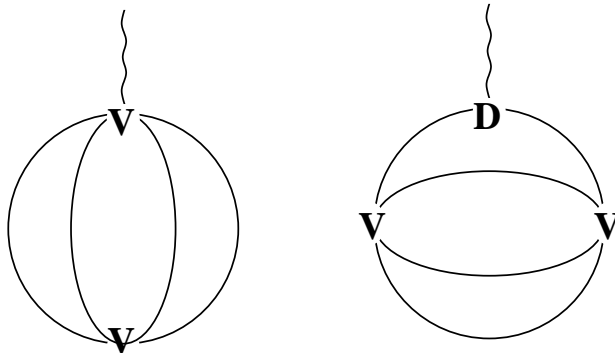


Figure 8: Contributions to the scalar stress-energy tensor at order  $\lambda^2$  with covariant normal ordering.  $V$  denotes the 4-point vertex and  $D$  represents the derivative vertex.

perturbation theory. We begin by evaluating the lowest order contribution to the expectation value of the stress-energy tensor. This result is then used to compute the expectation value of the invariant expansion operator which was defined in a previous paper [9]. The conclusion is that back-reaction slows inflation by an amount which eventually becomes nonperturbatively strong. All orders bounds are given for the strength of the effect. Finally, we argue that significant back-reaction would show up as well if stochastic samples, rather than expectation values, had been used.

With covariant normal-ordering the expectation value of the stress-energy tensor is much simpler than without. Because there are no coincident propagators the lowest contribution comes at order  $\lambda^2$  from the two diagrams in Fig. 8 [11]. (Of course there is a cosmological counterterm to absorb the ultraviolet divergence.) Further, the derivatives on the top vertex of the righthand diagram render its contribution subdominant to the lefthand diagram in powers of  $Ht$ . So the dominant contribution to the expectation value of the stress-energy tensor is  $-g_{\mu\nu}(x)$  times,

$$\begin{aligned} \frac{\lambda}{4!} \langle \Omega | : \phi^4(x) : | \Omega \rangle = \\ \frac{-i\lambda^2}{4!} \int_{t' > 0} d^4x' \Omega^4(\eta') \left\{ [i\Delta_{++}(x; x')]^4 - [i\Delta_{+-}(x; x')]^4 \right\} + O(\lambda^3). \end{aligned} \quad (58)$$

The difference of  $++$  and  $+-$  propagators comes from using the Schwinger-

Keldysh formalism [15, 29] to compute an expectation value rather than an in-out amplitude. This form ensures that the result is real and that it depends only upon points  $x'^\mu$  in the past lightcone of the observation point  $x^\mu$ . The lower limit of temporal integration at  $\eta' = -H^{-1}$  (that is,  $t' = 0$ ) derives from the fact that we release the state in free Bunch-Davies vacuum at this instant.

Although (58) was computed in ref. [11] we will go over it in detail. Since only the logarithm term of the propagator breaks conformal invariance it is perhaps not surprising that the dominant secular effect comes from taking this term in each of the four propagators. This contribution is completely ultraviolet finite, and its evaluation is straightforward if one goes after only the largest number of temporal logarithms,

$$\begin{aligned} & \frac{-i\lambda^2}{4!} \left( \frac{-H^2}{8\pi^2} \right)^4 \int_{-H^{-1}}^{\eta} d\eta' \left( \frac{-1}{H\eta'} \right)^4 4\pi \int_0^{\infty} dr r^2 \\ & \quad \times \left\{ \ln^4 \left[ H^2 (r^2 - (\Delta\eta - i\epsilon)^2) \right] - \ln^4 \left[ H^2 (r^2 - (\Delta\eta + i\epsilon)^2) \right] \right\} \\ & \rightarrow \frac{-i\lambda^2 H^4}{2^{13} 3^1 \pi^7} \int_{-H^{-1}}^{\eta} d\eta' \frac{1}{\eta'^4} \int_0^{\Delta\eta} dr r^2 8\pi i \ln^3 \left[ H^2 (\Delta\eta^2 - r^2) \right], \end{aligned} \quad (59)$$

$$= \frac{\lambda^2 H^4}{2^{10} 3^1 \pi^6} \int_{-H^{-1}}^{\eta} d\eta' \frac{\Delta\eta^3}{\eta'^4} \int_0^1 dx x^2 \left[ 2 \ln(H\Delta\eta) + \ln(1 - x^2) \right]^3, \quad (60)$$

$$\rightarrow \frac{\lambda^2 H^4}{2^7 3^2 \pi^6} \int_{-H^{-1}}^{\eta} d\eta' \frac{\Delta\eta^3}{\eta'^4} \ln^3(H\Delta\eta). \quad (61)$$

For large  $Ht$  the biggest effect comes from the term with the most factors of  $\ln(-H\eta) = -Ht$ . That the integrand contributes three such factors follows from the expansion,

$$\ln(H\Delta\eta) = \ln(-H\eta') - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta}{\eta'} \right)^n. \quad (62)$$

An additional factor comes from performing the integration up against the final term in the expansion of the ratio,

$$\frac{\Delta\eta^3}{\eta'^4} = \frac{\eta^3}{\eta'^4} - 3 \frac{\eta^2}{\eta'^3} + 3 \frac{\eta}{\eta'^2} - \frac{1}{\eta'}. \quad (63)$$

The final result is therefore,

$$\frac{\lambda}{4!} \langle \Omega | : \phi^4(x) : | \Omega \rangle = -\frac{\lambda^2 H^4}{2^9 3^2 \pi^6} \left\{ (Ht)^4 + O(H^3 t^3) \right\} + O(\lambda^3). \quad (64)$$

Three points deserve comment before we consider the effect on the invariant expansion observable. First, there is nothing paradoxical about the negative sign of the  $(Ht)^4$  contribution to expectation value of a positive operator. The actual result is dominated by a positive ultraviolet divergent constant. It is only after the cosmological counterterm is used to subtract this divergence that the ultraviolet finite factor of  $(Ht)^4$  dominates the late time behavior of the scalar stress-energy tensor at order  $\lambda^2$ .

Our second comment is that the negative sign of the  $(Ht)^4$  term has a simple physical interpretation. As the inflationary expansion rips more and more scalars out of the vacuum their attractive self-interaction acts to pull them back together. This is the direct analog of the graviton effect we have been seeking.

Our final comment is that the full stress-energy, including the bare cosmological constant, obeys the weak energy condition. Our leading order result implies,

$$\frac{\rho(t)}{H^4} = \frac{9}{8\pi G\Lambda} - \frac{\lambda^2}{2^9 3^2 \pi^6} \left\{ (Ht)^4 + O(H^3 t^3) \right\} + O(\lambda^3) , \quad (65)$$

$$\frac{p(t)}{H^4} = -\frac{9}{8\pi G\Lambda} + \frac{\lambda^2}{2^9 3^2 \pi^6} \left\{ (Ht)^4 + O(H^3 t^3) \right\} + O(\lambda^3) . \quad (66)$$

Since  $G\Lambda \ll 1$ , the sign of the total energy density is positive, even though the scalar contribution is negative. (At least for as long as perturbation theory remains valid.) From conservation we see that the sum of the energy density and the pressure is positive,

$$\rho(t) + p(t) = \frac{-\dot{\rho}(t)}{3H} = \frac{\lambda^2 H^4}{2^9 3^2 \pi^6} \left\{ \frac{4}{3} (Ht)^3 + O(H^2 t^2) \right\} + O(\lambda^3) . \quad (67)$$

This is the sense in which matter ought to act, so we conclude that covariant normal-ordering has succeeded in making scalars behave analogously to the vastly more complicated graviton model.

We have so far been working in a locally de Sitter geometry. To quantify the back-reaction on inflation we must instead regard de Sitter as the background upon which perturbative corrections are superimposed,

$$g_{\mu\nu}(x) \equiv \Omega^2(\eta) [\eta_{\mu\nu} + \kappa \psi_{\mu\nu}(x)] \quad , \quad \Omega(\eta) \equiv \frac{-1}{H\eta} . \quad (68)$$

Here  $\kappa^2 \equiv 16\pi G$ . The pseudo-graviton field,  $\psi_{\mu\nu}(x)$ , is determined, up to a diffeomorphism, by solving the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi GT_{\mu\nu}[g, \phi] . \quad (69)$$

Note that this need not entail quantizing gravity. It is perfectly consistent to suppress dynamical graviton degrees of freedom so that the pseudo-graviton field is only an operator through its dependence upon  $\phi$ , and this is what we shall do. One consequence is that  $\kappa\psi_{\mu\nu}$  receives its first nonzero contributions at order  $G$ . If these result in a secular reduction of the expansion rate — as we will see that they do — then we can establish that back-reaction is real without needing to consider either higher  $G$  corrections to  $\kappa\psi_{\mu\nu}$  or the effect of more than a single power of the pseudo-graviton field in the expansion operator.

For the purpose of quantifying back-reaction in this system it suffices to use the simplest of the invariant observables previously constructed for that purpose [9]. We first define a scalar measure of the expansion rate and then evaluate it on a geometrically fixed observation point. Our scalar is the inverse conformal d'Alembertian acting upon a unit source,

$$\mathcal{A}[g](x) \equiv \left( \frac{1}{\square_c} 1 \right) (x) , \quad (70)$$

$$= \mathcal{A}_0(x) + \kappa\mathcal{A}_1(x) + O(\kappa^2\psi^2) . \quad (71)$$

The zeroth order term can be evaluated exactly, although we shall make the slow roll approximation (denoted by an arrow),

$$\mathcal{A}_0(x) \equiv -\frac{1}{\Omega} \frac{1}{\partial^2} \Omega^3 , \quad (72)$$

$$= -e^{Ht} \int_0^t dt' e^{-Ht'} \int_0^{t'} dt'' e^{2Ht''} , \quad (73)$$

$$= -\frac{1}{2H^2} (1 - e^{-Ht})^2 , \quad (74)$$

$$\longrightarrow -\frac{1}{2H^2} . \quad (75)$$

Because this is constant, no perturbative shift in the observation point can affect the first order result! Therefore, it does not even matter how the observation point is geometrically determined although, for the sake of completeness, we employ zero shift with the following clock function to fix surfaces of

simultaneity, [9],

$$\mathcal{N}[g](x) \equiv -\left(\frac{1}{4\Box}R\right)(x) \longrightarrow Ht + O(\kappa\psi) . \quad (76)$$

The single graviton correction to  $\mathcal{A}[g](x)$  is simple to evaluate in the slow roll approximation [9],

$$\begin{aligned} \kappa\mathcal{A}_1(x) = -\frac{\kappa}{\Omega}\frac{1}{\partial^2} \left\{ -\psi^{\mu\nu}\partial_\mu\partial_\nu - (\psi^{\mu\nu}{}_{,\nu} - \frac{1}{2}\psi^{,\mu})\partial_\mu \right. \\ \left. - \frac{1}{6}(\psi^{\mu\nu}{}_{,\mu\nu} - \psi^{,\mu}{}_{,\mu}) \right\} \frac{1}{\partial^2}\Omega^3 , \end{aligned} \quad (77)$$

$$\longrightarrow \frac{1}{24H^4\Omega^2(\eta)} \left( \kappa\psi^{,\mu}{}_\mu(x) - \kappa\psi^{\mu\nu}{}_{,\mu\nu}(x) \right) . \quad (78)$$

This particular combination of the pseudo-graviton field happens to be fixed by the Einstein equation (69),

$$\kappa\mathcal{A}_1(x) \longrightarrow -\frac{\pi G}{3H^4}g^{\mu\nu}(x)T_{\mu\nu}(x) . \quad (79)$$

Since we need not consider corrections of higher order in  $G$ , the metric in this last expression can be set to the non-dynamical background,  $g_{\mu\nu}(x) \longrightarrow \Omega^2(\eta)\eta_{\mu\nu}$ . From the form of the stress-energy tensor (56), and our previous result for the leading contribution to its expectation value (64), we see that the slow roll approximation for the expectation value of the expansion operator is,

$$\langle\Omega|\mathcal{A}[g](x)|\Omega\rangle \longrightarrow -\frac{1}{2H^2} \left\{ 1 + \frac{\lambda^2 G\Lambda}{2^6 3^4 \pi^5} \left[ (Ht)^4 + O(H^3 t^3) \right] + O(\lambda^3, G^2) \right\} . \quad (80)$$

It follows that expectation values give the following estimate for the back-reacted expansion rate,

$$H_{\text{eff}}(x) = H \left\{ 1 - \frac{\lambda^2 G\Lambda}{2^7 3^4 \pi^5} \left[ (Ht)^4 + O(H^3 t^3) \right] + O(\lambda^3, G^2) \right\} . \quad (81)$$

This is precisely the same result that was previously obtained by studying the expectation value of the gauge fixed metric [11]. Back-reaction is for real.

Although we will not compute them here, it is straightforward to estimate the strength of higher order effects. Consider a diagram with  $2N$  external scalar lines. At  $L$  loop order the number of  $\phi^4$  interaction vertices is,

$$V = L + N - 1 . \quad (82)$$

Each contributes a factor of  $\lambda$ , so the diagram goes like  $\lambda^{L+N-1}$ . The number of internal propagators is,

$$P = 2L + N - 2 . \quad (83)$$

The largest secular effect comes, as it did for (64), from the term where each propagator contributes its logarithm part. Since we are computing a Schwinger diagram, there will be  $V$  cancellations between  $+$  and  $-$  variations, which give the  $\theta$ -function imaginary part of the logarithm. However, there are also  $V$  temporal integrations, each one of which can potentially result in an extra factor of  $\ln(-H\eta) = -Ht$ . Hence the strongest possible effect for the  $2N$ -point vertex at  $L$  loop order is,

$$V_{2N}^L \sim \lambda^{L+N-1} (Ht)^{2L+N-2} . \quad (84)$$

The stress-energy tensor corresponds to  $N = 0$  so the dominant contribution at  $L$  loop order is,

$$T_{\mu\nu}^L \sim g_{\mu\nu} H^4 \left( \lambda (Ht)^2 \right)^{L-1} . \quad (85)$$

It follows that perturbation theory breaks down at  $Ht \sim 1/\sqrt{\lambda}$ . Since  $\lambda$  is assumed small we see that back-reaction can be studied reliably for an enormous number of e-foldings. Note that all the higher point diagrams remain perturbatively weak during this entire period,

$$\lim_{Ht \rightarrow \lambda^{-1/2}} V_{2N}^L \sim \lambda^{N/2} . \quad (86)$$

It should therefore be valid to use perturbation theory almost up to  $Ht = 1/\sqrt{\lambda}$ .

Finally, we consider the effect of taking stochastic samples of  $\mathcal{A}[g](x)$  rather than computing its expectation value. A procedure for implementing this perturbatively was worked out in ref. [10]. What one does is to solve

the scalar field equations (57) for  $\phi(x)$  in terms of its initial value and that of its first derivative, organized as creation and annihilation operators on free Bunch-Davies vacuum,

$$\phi(x) = \phi_I(x) + \frac{\lambda}{6} \int_{t'>0} d^4x' G_{\text{ret}}(x; x') : \phi_I^3(x') : + O(\lambda^2) . \quad (87)$$

The free field  $\phi_I(x)$  was given in expression (34), and the retarded Green's function is,

$$G_{\text{ret}}(x; x') = -\frac{\theta(\Delta\eta)}{4\pi} \left\{ \frac{\delta(\Delta\eta - \Delta x)}{\Omega(\eta)\Omega(\eta')\Delta x} + H^2\theta(\Delta\eta - \Delta x) \right\} . \quad (88)$$

The order  $G$  result for the expansion observable comes from substituting this solution into (79) and then assigning random  $\mathbb{C}$ -number values to those creation and annihilation operators in  $\phi_I(x)$  which have experienced horizon crossing by the observation time.

Since the dominant contribution to the stress-energy tensor is from the quartic coupling we can make the replacement,

$$\kappa\mathcal{A}_1(x) \longrightarrow -\frac{\lambda G}{18H^4} : \phi^4(x) : . \quad (89)$$

The order  $\lambda$  contribution to this operator is obtained by replacing all the fields  $\phi(x)$  by the free field  $\phi_I(x)$ . Although this term has zero expectation value on account of covariant normal-ordering, a stochastic sample will generally be nonzero. We can compute its variance by taking the expectation value of its square,

$$\left\langle \Omega \left| \left( : \phi_I^4(x) : \right)^2 \right| \Omega \right\rangle = 24[i\Delta(x; x)]^4 . \quad (90)$$

Since  $i\Delta(x; x)$  grows like  $H^2/4\pi^2 Ht$ , we see that the order  $\lambda$  contributions to  $\kappa\mathcal{A}_1(x)$  fluctuate about zero with standard deviation,

$$\sigma_{\kappa\mathcal{A}_1} = \frac{1}{2H^2} \left\{ \frac{\lambda G \Lambda}{2^2 3^{2.5} \pi^3} [(Ht)^2 + O(Ht)] + O(\lambda^2, G^2) \right\} . \quad (91)$$

For some samples the fluctuation reduces the expansion rate from its classical value, for other samples the expansion rate is increased.

It is important to realize that this order  $\lambda$  effect is the same as the one previously studied on the nonperturbative level by Linde and collaborators

[8]. Although its variance does have the stated temporal dependence, what an actual observer sees is the effect of the field executing a drunkard's walk. Hence the time dependence of the order  $\lambda$  contribution to a stochastic sample of  $\mathcal{A}[g](x)$  is not simple.

The order  $\lambda^2$  effect we saw from the expectation value derives from the term where three of the fields in  $:\phi^4:$  are the free field  $\phi_I$  and the other field is the order  $\lambda$  correction in (87). The nonlocal character of this term tends to wash out its variance [10], so stochastic samples of it are clustered tightly around the mean value (80). We conclude that stochastic samples of the expansion operator consist of the same secular slowing term we found in its expectation value, superimposed upon a genuinely stochastic, random walk at order  $\lambda$ . Since there is only a small probability for the drunkard's walk to exhibit monotonic time dependence, it is possible to distinguish the two effects, even if a fluctuation happens to make the order  $\lambda$  contribution larger. Therefore, back-reaction is still real in the presence of stochastic effects.

## 6 Discussion

In this paper we have employed a simple scalar model to demonstrate that there can be significant back-reaction on inflation, even when the effect is quantified using an invariant operator measure of the expansion rate and even when stochastic effects are included. The expectation value of the invariant expansion operator gives precisely the same result that was previously inferred by computing the expectation value of the gauge-fixed metric [11]. The situation is more complicated when stochastic effects are included. The result in this case is that almost the same secular slowing is superimposed upon the perturbative analog of the stochastic effect previously studied by Linde and collaborators [8]. In both cases back-reaction slows inflation by an amount which eventually becomes nonperturbatively strong.

The scalar model is somewhat contrived in two ways. First, the scalar's natural mass is not zero but rather the scale of inflation,  $\sqrt{HM_{\text{pl}}}$ . Second, the covariant normal-ordering prescription of Section 4 results in a subdominant contribution to the stress-energy tensor (56) which depends causally but nonlocally upon the scalar and metric fields. This term plays no role at the order we worked, but it is necessary to enforce conservation at higher orders. Neither of these features should cast doubt upon the reality of back-reaction



in quantum gravity. In fact they were imposed upon the scalar model to make it more nearly resemble gravity.

In Section 2 we have also tried to answer the three objections of principle sometimes made to a significant back-reaction: causality, redshift, and averaging the source. To briefly recapitulate, the gravitational attraction between superhorizon particle pairs derives from virtual gravitons which were emitted in the past, before each particle exited its partner's causal horizon. The same mechanism is responsible for the persistence of gravitational fields due to massive objects which have fallen inside the event horizon of a black hole. Although electromagnetic fields are redshifted by inflation, gravitational potentials are not. This derives from the fact that electromagnetism is conformally invariant whereas gravity is not. On a concrete level one can see it from the theta function term in the retarded Green's function (88) which is common to minimally coupled scalars and dynamical gravitons. Finally, it is not valid to compute back-reaction from the average stress-energy of produced particles because this suppresses the quantum correlation between produced *pairs* and because the actual distribution of particles is not uniform. Clumps of energy density attract one another gravitationally whereas the perfectly uniform distribution which results from taking the spatial average simply increases the overall expansion rate.

A spinoff of our work is the order  $\lambda$  violation (49) of the weak energy condition when the scalar model is *not* covariantly normal-ordered. This seems to be an analog, on cosmological scales, of quantum violations of the energy conditions which have been previously studied on much smaller scales [19]. With observations on the current state of the universe not disfavoring an equation of state with  $w < -1$  [30] it is worth taking note of models that can achieve this.

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